Graduate Comprehensive Examination

Department of Mathematical Sciences

MA540, Probability and Mathematical Statistics I

January 19, 2017

1. Let $X, Y \overset{ind}{\sim}$ Gamma$(1,1)$ and $S = \frac{X}{p} - \frac{Y}{1-p}, 0 < p < 1$. Find the p^{th} quantile of S. What happens when $p = 1/2$?

2. Suppose $Z \sim$ Beta$(\alpha, \beta + \gamma)$. Show that Z can be written as $Z = XY$, where X and Y are independent with $X \sim$ Beta(α, β) and $Y \sim$ Beta$(\alpha + \beta, \gamma)$.

3. Let $Z \mid X = x, \delta \sim$ Normal$(\delta x, 1 - \delta^2), \mid \delta \mid < 1$ and $X \sim$ Normal$(0,1), x > 0$ (half normal). Show that $f(z) = 2\phi(z)\Phi(\lambda z)$, where $\lambda = \delta/\sqrt{1-\delta^2}$ and $\phi(\cdot)$ and $\Phi(\cdot)$ are respectively the pdf and cdf of the standard normal random variable. Find the moment generating function of Z.

4. Suppose that $X, Y|Z \sim$ Bernoulli(Z) and $Z \sim$ Beta(α, β).

(a) Find the expectation of X

(b) Find the Variance of X.

(c) Find covariance of X and Y.

(d) Show that X and Y are identically distributed.

5. Let $X_n \sim$ Poisson$(n\lambda)$ where the positive integer n is large and $0 < \lambda$.

(a) Find the limiting distribution of $\sqrt{n}\left(\frac{X_n}{n} - \lambda\right)$.

(b) Find the limiting distribution of $\sqrt{n}\left[\sqrt{\frac{X_n}{n}} - \sqrt{\lambda}\right]$.

6. Show that if U and V are independent uniform $(-1/2, 1/2)$ variables and $U^2 + V^2 \leq 1/4$, then U/V is a Cauchy variate.